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ON STABILITY OF SELF-CONTAINED HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM IN THE CASE OF ZERO FREQUENCIES*

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The stability problem for a self-contained Hamiltonian system with two degrees of freedom is solved for the case in which the fundamental equation of the linearized system has four zero roots.

1. Let us consider a self-contained Hamiltonian system with two degrees of freedom. Suppose the origin of the phase space corresponds to the equilibrium position of the system, and let the Hamiltonian function be analytic in some neighborhood of the equilibrium position, i.e.,

$$H = H_2 + \ldots + H_m + \ldots$$

where the H_m are homogeneous polynomials of degree m in generalized coordinates q_k and momenta p_k $(k = 1, 2)$:

$$H_m = \sum_{\nu_1+\nu_2+\mu_1+\mu_2=m} h_{\nu_1\nu_2\mu_1\mu_2} q_1^{\nu_1} q_2^{\nu_2} p_1^{\mu_1} p_2^{\mu_2}$$

The Liapunov stability of such systems in a rigorous nonlinear formulation has now been studied for all possible cases (see /1,2/). other than the case of the fundamental equation with four zero roots (i.e., the case of two zero frequencies, or double first-order resonance). The present paper is devoted to a solution of this problem. As an illustration, we consider the converse to the Lagrange– Dirichlet theorem.

In accordance with procedures developed for studying all the earlier cases, first we consider normalization of the linearized system corresponding to the quadratic part of the Hamiltonian function. For this purpose, we write the linearized system in the form

$$dx / dt = Jhx, \quad x = (q_1, q_2, p_1, p_2)^T \tag{1.1}$$

$$J = -J^T = \left\| \begin{matrix} O_2 & E_2 \\ -E_2 & O_2 \end{matrix} \right\|, \quad h = \left\| \frac{\partial^2 H_2}{\partial x^2} \right\|$$

where O, E are the zero and unitary matrices of corresponding orders. Then the normalization problem reduces to finding a nondegenerate, real, simplectic matrix N, such that the transformation

$$x = Nx', \quad x' = (q_1', q_2', p_1', p_2')^T \tag{1.2}$$

reduces the linear system (1.1) to the form

$$dx'/dt = Jh'x', \quad h' = \left\| \frac{\partial^2 H_2'}{\partial x'^2} \right\| \tag{1.3}$$

In fact normal forms for the quadratic Hamiltonians H_2' for all possible types of eigenvalues of the matrix Jh were found by Williamson (see Appendix to /3/), i.e., the relation between H_2' and q_k', p_k' is known. In our problem, in which all the eigenvalues of Jh are zero, depending on the rank of h the following cases may arise (a more complicated normal form has been presented /3/ for the case of general position $\operatorname{rg} h = 3$):

$$H_2' = {}^1/_2\delta p_1'^2 - q_1'q_2' \quad (\delta = \pm 1), \quad \operatorname{rg} h = 3 \tag{1.4}$$

$$H_2' = {}^1/_2\delta_1 p_1'^2 + {}^1/_2\delta_2 p_2'^2 \quad (\delta_1 = \pm 1,\ \delta_2 = \pm 1), \quad \operatorname{rg} h = 2 \tag{1.5}$$

$$H_2' = {}^1/_2\delta p_1'^2 \quad (\delta = \pm 1), \quad \operatorname{rg} h = 1 \tag{1.6}$$

$$H_2' \equiv 0, \quad \operatorname{rg} h = 0 \tag{1.7}$$

Note that there is as yet no normalization algorithms for the case of two or more zero frequencies. We propose here a constructive algorithm which be used to find normalizing transformation matrices for all possible cases; our algorithm is also simpler than those previously

*Prikl.Matem.Mekhan.,45,No.3,441-449,1981 presented in the literature, (The complete set of linear normalization methods may be found in /1/ (*).

The required matrix must first reduce the matrix Jh to the form Jh' i.e., $JhN = NJh'$, second, it must be simplectic, i.e.,

$$N^T JN = J \tag{1.8}$$

The solution of the first equation exists if and only if the matrices Jh and Jh' have identical normal Jordan form /4/. Let G is the normal Jordan form of these matrices. Clearly, the matrix which reduces Jh to the normal Jordan form will not, in general, be simplectic, if for no other reason that only Jordan cells of order no higher than the first (with an even number of cells) may correspond to the same canonical system. However, the product of two nonsimplectic matrices may turn out to be a simplectic matrix. Accordingly, we seek the normalizing transformation matrix in the form $N = AB$. Here A is an arbitrary matrix which reduces Jh to the normal Jordan form, i.e., it is an arbitrary (but fixed) nondegenerate solution of the equation $JhA = AG$ composed of eigenvectors and adjoint vectors $\mathbf{a}_j$ of the matrix Jh. The matrix $B = C^{-1}$, where the matrix C reduces Jh' to the same Jordan form G: $Jh'C = CG$. In compiling C from the eigenvectors and adjoint vectors of Jh', we retain all the arbitrary constants which normalize these vectors. Note that the matrices B may be found in advance for all known sets of eigenvalues. The obtained arbitrariness may now be used for obtaining N as a simplectic matrix.

From (1.8) we also have the normalization relation

$$B^T FB = J \tag{1.9}$$

where $F = A^T JA$ is a skew-symmetric matrix, since $f_{jn} = (\mathbf{a}_j, J\mathbf{a}_n) = (J^T\mathbf{a}_j, \mathbf{a}_n) = -(\mathbf{a}_n, J\mathbf{a}_j) = -f_{nj}$. Further study of the structure of F is most easily performed for each case separately, in precisely the same way as in the case of simple eigenvalues (see /1/). Let us apply this simple idea to our problem.

In the case $\operatorname{rg} h = 3$, we have

$$\begin{aligned} Jh\mathbf{a}_1 &= 0,\\ Jh\mathbf{a}_2 &= \mathbf{a}_1,\\ Jh\mathbf{a}_3 &= \mathbf{a}_2,\\ Jh\mathbf{a}_4 &= \mathbf{a}_3, \end{aligned} \qquad B = \begin{Vmatrix} \delta b_2 & b_4 & b_3 & \delta b_1\\ \delta b_1 & b_3 & b_2 & 0\\ 0 & b_2 & b_1 & 0\\ 0 & b_1 & 0 & 0 \end{Vmatrix}, \quad F = \begin{Vmatrix} 0 & 0 & 0 & f_{14}\\ 0 & 0 & -f_{14} & 0\\ 0 & f_{14} & 0 & f_{34}\\ -f_{14} & 0 & -f_{34} & 0 \end{Vmatrix} \tag{1.10}$$

where the b_j are arbitrary real numbers $(b_1 \neq 0)$, and $f_{14} \neq 0$, since $f_{14}^4 = \det F = (\det A)^2 \neq 0$. Substituting the expressions for B and F in the normalization relation (1.9), we obtain equations for δ and b_j:

$$\delta b_1^2 f_{14} = -1, \quad 2b_1 b_3 f_{14} - b_2^2 f_{14} + b_1^2 f_{34} = 0$$

Setting, the sake of simplicity, $b_2 = b_4 = 0$, we obtain the final expression for the normalizing matrix:

$$\begin{gathered} N = \| \delta b_1\mathbf{a}_2,\ b_3\mathbf{a}_2 + b_1\mathbf{a}_4,\ b_3\mathbf{a}_1 + b_1\mathbf{a}_3,\ \delta b_1\mathbf{a}_1 \|\\ \delta = -\operatorname{sign}(\mathbf{a}_1, J\mathbf{a}_4),\ b_1 = |(\mathbf{a}_1, J\mathbf{a}_4)|^{-1/2}, \quad b_3 = {}^1/_2\delta b_1^3(\mathbf{a}_3, J\mathbf{a}_4) \end{gathered}$$

where $\mathbf{a}_j$ $(j = 1, 2, 3, 4)$ are arbitrary linearly independent solutions of equations (1.10).

For the case $\operatorname{rg} h = 2$ we find

$$\begin{aligned} Jh\mathbf{a}_1 &= 0,\\ Jh\mathbf{a}_2 &= \mathbf{a}_1,\\ Jh\mathbf{a}_3 &= 0,\\ Jh\mathbf{a}_4 &= \mathbf{a}_3, \end{aligned} \qquad B = \begin{Vmatrix} b_1 & 0 & b_3 & 0\\ 0 & 0 & \delta_1 b_1 & 0\\ 0 & b_2 & 0 & b_4\\ 0 & 0 & 0 & \delta_2 b_2 \end{Vmatrix}, \quad F = \begin{Vmatrix} 0 & f_{12} & 0 & 0\\ -f_{12} & 0 & 0 & 0\\ 0 & 0 & 0 & f_{34}\\ 0 & 0 & -f_{34} & 0 \end{Vmatrix} \tag{1.11}$$

where $f_{12} \neq 0$, $f_{34} \neq 0$, since $f_{12}^2 f_{34}^2 = \det F = (\det A)^2 \neq 0$. As in the preceding case we find from (1.9) and (1.11) equations for δ_1, δ_2 and b_j

$$\delta_1 b_1^2 f_{12} = 1, \quad \delta_2 b_2^2 f_{34} = 1$$

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*) See also Titova, T.N., Normalization of Hamiltonian Matrices. Dissertation Presented to the Senior School Candidate Competition in Physics and Mathematics. Moscow, Peoples Friendship University, 1978.

Setting, for simplicity, $b_3 = b_4 = 0$ we have the final expression

$$N = \| b_1\mathbf{a}_1,\ \ b_2\mathbf{a}_3,\ \ \delta_1 b_1\mathbf{a}_2,\ \ \delta_2 b_2\mathbf{a}_4 \|$$
$$\delta_1 = \operatorname{sign}(\mathbf{a}_1, J\mathbf{a}_2),\ b_1 = |(\mathbf{a}_1, J\mathbf{a}_2)|^{-1/2},\ \delta_2 = \operatorname{sign}(\mathbf{a}_3, J\mathbf{a}_4),$$
$$b_2 = |(\mathbf{a}_3, J\mathbf{a}_4)|^{-1/2}$$

where $\mathbf{a}_j$ is the solution of equations (1.11).
Finally, for the case $\operatorname{rg} h = 1$

$$\begin{aligned} Jh\mathbf{a}_1 &= 0,\\ Jh\mathbf{a}_2 &= \mathbf{a}_1,\\ Jh\mathbf{a}_3 &= 0,\\ Jh\mathbf{a}_4 &= 0,\end{aligned}\qquad B = \begin{Vmatrix} b_1 & 0 & b_2 & 0\\ 0 & 0 & \delta b_1 & 0\\ 0 & b_3 & 0 & 0\\ 0 & 0 & 0 & b_4\end{Vmatrix},\quad F = \begin{Vmatrix} 0 & f_{12} & 0 & 0\\ -f_{12} & 0 & 0 & 0\\ 0 & 0 & 0 & f_{34}\\ 0 & 0 & -f_{34} & 0\end{Vmatrix} \qquad (1.12)$$

and, as in the preceding case, $f_{12} \neq 0,\ f_{34} \neq 0$. Using the normalization condition, we obtain the final expression for the normalizing matrix:

$$N = \| b_1\mathbf{a}_1,\ \ b_3\mathbf{a}_3,\ \ \delta b_1\mathbf{a}_2,\ \ \mathbf{a}_4 \|$$
$$\delta = \operatorname{sign}(\mathbf{a}_1, J\mathbf{a}_2),\ b_1 = |(\mathbf{a}_1, J\mathbf{a}_2)|^{-1/2},\ b_2 = 0$$
$$b_3 = (\mathbf{a}_3, J\mathbf{a}_4)^{-1},\ b_4 = 1$$

In the case $\operatorname{rg} h = 0$ normalization need not be performed, since it follows from the very condition $\operatorname{rg} h = 0$ that all the coefficients of the quadratic part of the Hamiltonian are zero, further, and it is already of normal form (1.7).

We further assume that a linear normalization has already been performed in the system and that the quadratic part of the Hamiltonian function has the form (1.4)—(1.7) for the cases $\operatorname{rg} h = 3, 2, 1, 0$, respectively. Previous notation (without primes) will be used for the phase variables.

2. Let us consider the stability problem for a complete nonlinear system in the general position case $\operatorname{rg} h = 3$. For this purpose, in the complete system we perform using the De Pree-Hori method, a nonlinear normalization (*) $(q_k, p_k) \to (Q_k, P_k)$ $(k = 1, 2)$ by which the new Hamiltonian function $K = K_2 + \ldots + K_m + \ldots$ assumes a simpler form. By $S = S_3 + \ldots + S_m + \ldots$ we denote the generating function of the De Pree—Hori method, obtaining for the coefficients $s_{\nu_1\nu_2\mu_1\mu_2}$ of its forms S_m and the coefficients $k_{\nu_1\nu_2\mu_1\mu_2}$ of the new Hamiltonian the system of algebraic equations

$$\delta(\nu_1 + 1)\, s_{\nu_1+1,\, \nu_2,\, \mu_1-1,\, \mu_2} - (\mu_1 + 1)\, s_{\nu_1,\, \nu_2-1,\, \mu_1+1,\, \mu_2} - (\mu_2 + 1)\, s_{\nu_1-1,\, \nu_2,\, \mu_1,\, \mu_2+1} = g_{\nu_1\nu_2\mu_1\mu_2} - k_{\nu_1\nu_2\mu_1\mu_2} \qquad (2.1)$$
$$(\nu_1 + \nu_2 + \mu_1 + \mu_2 = m;\ m = 3, 4, \ldots)$$

where $g_{\nu_1\nu_2\mu_1\mu_2}$ are the coefficients of the forms G_m expressed in terms of the forms S_n, H_n, K_n of lower orders; for example, $G_3 = H_3$, $G_4 = H_4 + {}^1/_2 \{S_3,\ H_3 + K_3\}$ ($\{\ ,\ \}$ are the Poisson brackets). The solution of equations (2.1) yields a normal form for the Hamiltonian function (through third-order terms):

$$K = K^{(0)} + K^{(1)} \qquad (2.2)$$
$$K^{(0)} = \frac{1}{2}\delta P_1^2 - Q_1 Q_2 + k_{0003} P_2^3 \quad (k_{0003} = h_{0003}) \qquad (2.3)$$
$$K^{(1)} = k_{0102} Q_2 P_2^2 + k_{0012} P_1 P_2^2 + K_4 + \ldots$$

Theorem 2.1. If $k_{0003} \neq 0$, the equilibrium position is unstable.

To prove the theorem, we will first consider a truncated system with Hamiltonian function (2.3). It has the unstable particular solution

$$Q_1 = aP_2^{5/4},\quad P_1 = bP_2^{7/4},\quad Q_2 = cP_2^{7/4},\quad P_2 = P_2(0)[1 - At]^{-4} \qquad (2.4)$$
$$a = 4A\,[P_2(0)]^{-1/4},\quad b = 20\delta A^2\,[P_2(0)]^{-2/4},$$
$$c = 120\delta A^3\,[P_2(0)]^{-3/4}$$
$$A = [\delta k_{0003} P_2(0)\,/\,280]^{1/4}$$

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*) see Markeev, A.P. and Sokol'skii A.G., "Certain computational normalization algorithms for Hamiltonian systems. Preprint, Inst. Prikl. Matem. Akad. Nauk SSSR, No.31, 1976.

Note that the solution we have found for the system with Hamiltonian (2.3) infinitely increases in the finite time $t \sim [P_2(0)]^{-1/4}$ under arbitrarily infinitesimal initial conditions $P_2(0)$, while the solutions of the linear system with Hamiltonian (1.4) may increase only by a power law. Using this unstable particular solution of the truncated system and the Chetaev theorem /5/ let us prove that the complete system is unstable. As an example, we take

$$V = P_2^{210} - \left[\left(\frac{Q_1}{a}\right)^4 - P_2^5\right]^{42} - \left[\left(\frac{P_1}{b}\right)^2 - P_2^3\right]^{70} - \left[\left(\frac{Q_2}{c}\right)^4 - P_2^7\right]^{30} \tag{2.5}$$

as the Chetaev function.

In the region $V > 0$ the following estimates are valid:

$$Q_1 = a(1+\alpha^5)^{1/4} P_2^{5/4},\ P_1 = |b|(1+\beta^3)^{1/2} P_2^{3/2},\ Q_2 = |c|(1+\gamma^7)^{1/4} P_2^{7/4}$$
$$P_2 > 0,\quad 0 < \alpha^{210} + \beta^{210} + \gamma^{210} < 1,\quad \alpha > 0,\quad \beta > 0,\quad \gamma > 0$$

It can be verified that the derivative of function V generated by means of equations of motion with Hamiltonian (2.2) are be positive definite in the region $V > 0$. Consequently, by the Chetaev theorem, the equilibrium position is unstable and Theorem 2.1 is proved.

Let us briefly discuss the degenerate case $k_{0003} = 0$. Normalization in this case must be performed through fourth-order terms, and the normal form now appears as (2.2), where

$$K^{(0)} = \frac{1}{2}\delta P_1^2 - Q_1 Q_2 + k_{0012} P_1 P_2^2 + k_{0004} P_2^4 \tag{2.6}$$
$$K^{(1)} = k_{0102} Q_2 P_2^2 + k_{0103} Q_2 P_2^3 + k_{0022} P_1^2 P_2^2 + k_{0013} P_1 P_2^3 + K_5 + \ldots$$

We set $d = k_{0012}^2 + 6\delta k_{0004}$, and suppose $d > 0$. Then the truncated system with Hamiltonian (2.6) has a particular solution analogous to the solution (2.4)

$$Q_1 = aP_2^{3/2},\quad P_1 = bP_2^{4/2},\quad Q_2 = cP_2^{5/2},\quad P_2 = P_2(0)[1 - At]^{-2}$$
$$a = 2A[P_2(0)]^{-1/2},\quad b = {}^1/_3\delta[-k_{0012} \pm \sqrt{d}]$$
$$c = 4bA[P_2(0)]^{-1/2}$$
$$A = \{P_2(0)[2k_{0012} \pm \sqrt{d}]/6\}^{1/2}$$

As in the nondegenerate case $k_{0003} \neq 0$, therefore, we find that the equilibrium position is unstable when $d > 0$. When $d < 0$ there is no analogous increasing solution of the truncated system and, apparently, the equilibrium position is Liapunov-stable. However, a rigorous proof of this assertion is not possible, since even the truncated system has no integral other than $K^{(0)}$ which is analytic near zero.

As an example of the application of the results of Sect.2 to actual mechanical problems, we consider the stability problem for the conical precession of a dynamically symmetric satellite in circular orbit /6/. Suppose $\alpha = 4/3, \beta = 0$ where α is the ratio of the polar and equatorial moments of inertia of the satellite, and β is the ratio of the projection of the absolute angular velocity of the satellite on its axis of symmetry and the angular velocity of the center of mass /7/. With these parameters, the satellite will move forward into absolute space, with its axis of symmetry perpendicular to the velocity vector of the center of mass, forming an arbitrary angle ϑ_0 with the normal to the orbital plane.

In this case, the first terms of the expansion of the Hamiltonian function of perturbed motion in corresponding coordinates have the form /7/

$$H_2 = p_1^2/(2s^2) + p_2^2/2 - q_1^2 s^2/2 + 2c^2 q_2^2 - 2p_1 q_2 c/s \tag{2.7}$$
$$H_3 = -2q_2^3 c/s + q_2^2 p_1(1+2c^2)/s^2 - q_2 p_1^2 c/s^3 - csq_1^2 q_2 \tag{2.8}$$
$$c = \cos\vartheta_0,\ s = \sin\vartheta_0$$

The case $\vartheta_0 = \pi/3$ in which the fundamental equation of the linear system with Hamiltonian function (2.7) and $\sigma^4 + (4c^2 - 1)\sigma^2 = 0$ has four zero roots, while the rank of the corresponding matrix h is three was not studied in an earlier review /7/ of this problem. Using the algorithm of Sect.1, we may find the linear normalizing transformation matrix:

$$N = \begin{Vmatrix} 2/\sqrt{3} & 1/\sqrt{3} & 0 & 0 \\ 0 & 0 & -1/2 & 1 \\ 0 & 0 & \sqrt{3}/4 & \sqrt{3}/2 \\ 1 & -1/2 & 0 & 0 \end{Vmatrix},\ \delta = 1$$

Passing to new variables in (2.8), we find the coefficient of the normal form (2.3) $k_{0003} = 1/\sqrt{3} \neq 0$. It follows from Theorem 2.1 that the conical precession is unstable.

Still another interesting example of the application of our results has been presented previously(*). It also turned out that $\omega_1 = \omega_2 = 0$ and $\operatorname{rg} h = 3$ in a study of the stability of the cylindrical precession of a symmetric satellite with values of the parameters $\alpha = 2/3$, $\beta = 3/2$. Reduction to normal form showed that in (2.3) $\delta = 1$, $k_{0003} = 0$, i.e., here we are dealing with a degenerate case. Further computations yield $k_{0012} = 0$ and $k_{0004} = 1/8$. Consequently, $d = k_{0012}^2 + 6\,\delta k_{0004} > 0$, so that on the basis of the foregoing we are led to conclude that cylindrical precession is unstable.

3. Let us consider the stability problem in the case $\operatorname{rg} h = 2$. In this case, to noramalize the nonlinear terms we use in place of equations (2.1) the following equations for determining the coefficients of the generating function and the coefficients of the new Hamiltonian function:

$$\delta_1 (\nu_1 + 1)\, s_{\nu_1+1,\, \nu_2,\, \mu_1-1,\, \mu_2} + \delta_2 (\nu_2 + 1)\, s_{\nu_1,\, \nu_2+1,\, \mu_1,\, \mu_2-1} = g_{\nu_1\nu_2\mu_1\mu_2} - k_{\nu_1\nu_2\mu_1\mu_2} \tag{3.1}$$

The solution of these equations yields a normal form for the Hamiltonian function (through third-order terms):

$$K = K^{(0)} + K^{(1)}, \quad K^{(0)} = K_2 + K_3^{(0)}, \quad K^{(1)} = K_3^{(1)} + K_4 + \ldots \tag{3.2}$$

$$K_3^{(0)} = h_{30} Q_1^3 + h_{21} Q_1^2 Q_2 + h_{12} Q_1 Q_2^2 + h_{03} Q_2^3 \quad (h_{\nu_1\nu_2} = h_{\nu_1\nu_2 00} = k_{\nu_1\nu_2 00}) \tag{3.3}$$

$$K_3^{(1)} = h_{1110} Q_1 Q_2 P_1 + h_{1101} Q_1 Q_2 P_2 \tag{3.4}$$

Theorem 3.1. If $h_{30}^2 + h_{21}^2 + h_{12}^2 + h_{03}^2 \neq 0$, the equilibrium position is unstable.

The theorem may be proved in the same way as the assertions of Sect.2. For this purpose, note that when the condition of the theorem is satisifed (i.e., when at least one of the coefficients of the form $K_3^{(0)}$ is nonzero), the truncated system with Hamiltonian $K^{(0)}$ admits of a particular solution of the form

$$Q_1 = \frac{Q_1(0)}{(1 - At)^2}, \quad Q_2 = \frac{Q_2(0)}{(1 - At)^2}, \quad P_1 = \frac{2\delta_1 A Q_1(0)}{(1 - At)^3} \tag{3.5}$$

$$P_2 = \frac{2\delta_2 A Q_2(0)}{(1 - At)^3}$$

$$A = \{-\delta_1 [3h_{30} Q_1^2(0) + 2h_{21} Q_1(0) Q_2(0) + h_{12} Q_2^2(0)] / [6Q_1(0)]\}^{1/2} = \{-\delta_2 [h_{21} Q_1^2(0) + 2h_{12} Q_1(0) Q_2(0) + 3h_{03} Q_2^2(0)] / [6Q_2(0)]\}^{1/2}$$

where $Q_1(0)$, $Q_2(0)$ are any simultaneously nonzero real numbers (they always exist; if, for example, $h_{21} = 0$, we take $Q_2(0) = 0$ and $Q_1(0)$ will be an arbitrary nonzero number) that satisfy the relation

$$\delta_1 h_{12} Q_2^3(0) + [2\delta_1 h_{21} - 3\delta_2 h_{03}] Q_2^2(0) Q_1(0) - [2\delta_2 h_{12} - 3\delta_1 h_{30}] Q_2(0) Q_1^2(0) - \delta_2 h_{21} Q_1^3(0) = 0$$

Using this increasing solution of the truncated system and selecting $Q_1(0)$, $Q_2(0)$ as small enough numbers, a Chetaev function of the complete system may be constructed analogous to the function (2.5).

If the condition of Theorem 3.1 does not hold, normalization must be performed through terms of higher order, and in the general case, the procedure becomes highly complicated (see Sect. 4) due to the presence of terms of the form $Q_1 Q_2 P_1$, $Q_1 Q_2 P_2, \ldots$ (proportional to P_k).

The stability problem for the cases $\operatorname{rg} h = 1$ and $\operatorname{rg} h = 0$ discussed above may be solved by combining the results of the present paper and previous results /2/.

4. As an example of the use of earlier results, we will briefly consider the relation of these results to the well-known converse problem to the Lagrange–Dirichlet stability theorem for a two-dimensioal conservative system. Note that this problem has been nearly completely solved /5, 8—11/ (see also /12, 13/). However, the most complete recent results /9—11/ were obtained by means of methods from optimal control theory and topology and do not have an explicit mechanical meaning. We would therefore like to solve this problem of mechanics using methods of analytic mechanics exclusively.

Suppose we are given a conservative mechanical system with two degrees of freedom, where q_1, q_2 are its Lagrangian coordinates and p_1, p_2 the corresponding generalized momenta; the coordinate origin of the phase space is an isolated equilibrium position. The kinetic energy

*) Sokol'skii, A.G. Stability problem for regular precessions of a symmetric satellite.

$$T = {}^1/_2 \sum_{j,k=1}^{2} [\delta_{jk} + \tau_{jk}(q_1, q_2)]\, p_j p_k; \quad \tau_{jk} = \tau_{kj},\ \tau_{j,k}(0, 0) = 0$$

where δ_{jk} is the Kronecker symbol, is a positive definite quadratic momentum form. Since the system is conservative, the forces affecting it are potential forces, while the potential energy (which is an analytic function of the Lagrangian coordinates in a neighborhood of the equilibrium position) has the form

$$U(q_1, q_2) = U_2 + U_3 + \ldots, \quad U_m = \sum_{j+k=m} u_{jk} q_1^j q_2^k \tag{4.1}$$

The perturbed motion equations may be written in canonical form with the Hamiltonian function $H = T + U$. In this form, a Hamiltonian system possesses an important property that distinguishes it from the general class of self-contained Hamiltonian systems (which, in general, may include gyroscopic forces) and also helps in its study. That is, the generalized momenta occur in the Hamiltonian function only quadratically. Under these conditions, the following assertion is true.

Theorem (Lagrange—Dirichlet). The equilibrium position of the above system is stable if and only if the potential energy $U(q_1, q_2)$ has a minimum at the equilibrium position, i.e., is a positive definite function of its variables q_1, q_2 in a neighborhood of the equilibrium position.

The first part of this assertion constitutes the content of the original Lagrange—Dirichlet theorem proper, while the second part has been referred to as the converse to the Lagrange—Dirichlet theorem and has remained unproved for some time.

Let us first consider the linearized system, writing it, as in /12/, in principal coordinates (without changing the notation for the variables and noting that the quadratic dependence of the Hamiltonian on the momenta is preserved). The Hamiltonian of such a linear system has the form

$$H_2 = {}^1/_2 p_1^2 + {}^1/_2 p_2^2 + {}^1/_2 C_1 q_1^2 + {}^1/_2 C_2 q_2^2 \quad (U_2 = {}^1/_2 C_1 q_1^2 + {}^1/_2 C_2 q_2^2,\ C_1 \geqslant C_2)$$

Note that $C_k = -\sigma_k^2\ (k = 1,2)$, where σ_k are the roots of the fundamental equation.

The following cases are possible: (a) $C_2 < 0$; (b) $C_1 = C_2 = 0$; (c) $C_1 > C_2 = 0$; (d) $C_1 \geqslant C_2 > 0$. For cases (b)—(d), we introduce the notation $C_k = \omega_k^2$, where ω_k are the frequencies of the linear oscillations.

In case (a), the linear system is unstable due to the presence in the general solution of terms which increase exponentially with time t, i.e., the complete system is also unstable; in the two cases (b) and (c), the linear system is unstable due to the presence of terms proportional to $t^n\ (n = 1, 2, 3)$ in the general solution, though we still cannot yet conclude that the complete system is unstable; and in case (d), both the linear and the complete systems are stable, as follows from the Liapunov stability theorem /14/ if we take a fixed-sign (in this case, positive definite) integral $H = \text{const}$ as the Liapunov function.

On the other hand, in case (a), the form U_2 is either negative definite $(C_2 \leqslant C_1 < 0)$, or is of negative sign $(C_1 = 0 > C_2)$, or alternates in sign $(C_1 > 0 > C_2)$,, i.e., the entire function (4.1) is nowhere positive definite. In case (d), U_2 (that is, the entire function (4.1)) is positive definite. In case (c), U_2 has positive terms, i.e., depending on $U_3, U_4, \ldots$ (4.1) may be either positive definite or alternate in sign. In case (b), $U_2 \equiv 0$ and is fully determined by $U_3, U_4, \ldots$.

Thus, in our study of stability cases (b) and (c) are special cases in which the forms $H_3, H_4, \ldots$ must be taken into account in the expansion of the Hamiltonian. In other words, we must consider the cases of one or two zero frequencies.

Case (c) (the case of a single zero frequency $\omega_2 = 0, \omega_1 \neq 0$) was considered quite exhaustively in /2/, which studied the stability of an arbitrary Hamiltonian system (i.e., gyroscopic forces are possible in the corresponding mechanical system, or the system is a generalized conservative system). In order to use these results, we need only make a single change in the proof of the corresponding Theorem 4.1 /2/: it is not the entire Hamiltonian function $H = T + U$ which is normalized, but rather only its "potential part" U. Then the normal form of the Hamiltonian is

$$K = ({}^1/_2 P_1^2 + {}^1/_2 \omega_1^2 Q_1^2) + ({}^1/_2 P_2^2 + a_{0,M} Q_2^M) + K^{(M)} + K_{M+1} + \cdots$$

where $a_{0,M} \neq 0$ while $K^{(M)}$ gathers together all terms of order no greater than M in Q_k, P_k, but such that their order is greater than $2M$ in ε under the substitution $P_1 = \varepsilon^M P_1^*$, $Q_1 = \varepsilon^M Q_1^*$, $P_2 = \varepsilon^M P_2^*$, $Q_2 = \varepsilon^2 Q_2^*$ /2/. Then by Theorem 4.1 /2/, we find that stability will hold only if M

is an even number and $a_{0,M} > 0$. But if M is an odd number, or if M is even but $a_{0,M} < 0$, the equilibrium position is unstable. Clearly, the stability condition coincides with the fixed-sign condition on (4.1). Thus, the only case we have not studied when $\omega_2 = 0$ is the so-called transcendental case, in which $a_{0,M} = 0$ for all $M = 3,4,\ldots$ (such a situation is found when, for example, the coordinate q_2 is an ignorable coordinate). However, we cannot conclude that the function (4.1) has (or does not have) an extremum in this case.

Finally, let us consider case (b): $\omega_1 = \omega_2 = 0$, using the results of Sect. 3 of the present paper.

Clearly, Theorem 3.1 is entirely in agreement with the assertions of the Lagrange—Dirichlet theorem we have been considering, since any third-order term (and any analytic function whose expansion starts with this form) is of alternating sign. Further, in (3.2), the terms (3.4) are missing in the case of a conservative system due to the quadratic dependence of the Hamiltonian on the momenta. It is therefore difficult to extend Theorem 3.1 to the case in which the expansion of (4.1) starts with any form U_m of odd degree m or in which m is an even number, but U_m is of alternating sign or negative definite (has negative terms). In all these cases, it is possible to find an unstable particular solution (such as (3.5)) of the truncated system, and then construct a Chetaev function of the form of (2.5) for the complete system. The procedure will be more complicated if the expansion of (4.1) starts with the form U_m of positive sign. In this case, the function $K^{(0)}$ from (3.2) must include not only U_m, but also terms from forms U_n of higher order, which results in the function becoming either positive definite (then, by the Liapunov theorem, the equilibrium position is stable) or of alternating sign. In the latter case, instability will occur, though this can be proved not by finding particular solutions such as (3.5), but rather, as in /15/, by studying the case $\omega_1 = 3\omega_2$ and $|c_{20} + 3c_{11} + 9c_{02}| = 3[3(A_{13}^2 + B_{13}^2)]^{1/2}$.

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